Ilya Gikman and John Sasser

Ilya Gikhman, Ph.D.
Institute of Applied Mathematics
Donetsk, Ukraine
and
Ukrainian
Academy of Sciences

AND

John E. Sasser, Ph.D.
International Eurasian
Academy of Sciences

The Solutions to Stochastic Differential Equations on Smooth Stable Manifolds

This article deals with the study of asymptotic behavior of the solution of an ordinary stochastic differential equation (SDE) when time increases to infinity.

We will prove that under certain assumptions on coefficients the solution of Ito's equation is attracted to the given manifold.

Using Ito Calculus, a set of ordinary differential equations may be determined that will describe the moments of a random process. In the Ito calculus, there are two different types of differential elements. There are *dt* terms, which are small; and there are $d\beta$ terms (Brownian motion terms), which are random. Brownian motion is the integral of *white noise*; i.e., $\beta(t)=\int_0^t n(s)\,ds$, when n(s) is white noise. We assume the standard scaling: $E[(d\beta)^2] = dt$, where the E[•] operator represents an expectation taken over the random variables in the system. The Brownian motion terms also have mean zero: $E[d\beta] = 0$. Suppose that $x_1(t)$ and $x_2(t)$ are random processes described by the two stochastic differential equations:

$$\frac{dx_1}{dt} = a_1(t) + b_1(t)n(t),$$
$$\frac{dx_2}{dt} = a_2(t) + b_2(t)n(t),$$

or

$$dx_1 = a_1(t) + b_1(t)\,d\beta,$$
$$dx_2 = a_2(t) + b_2(t)\,d\beta.$$

Ito's lemma states that:

$$d(x_1x_2) = x_1dx_2 + x_2dx_1 + b_1b_2dt$$

This relation is different from the result in the classical calculus by the inclusion of the last term. This relationship is used to determine moment equations for a random process.

Thus, the problem:

<u>To prove that under certain assumptions on coefficients the solution of Ito's equation is attracted to the given manifold</u>,

can be interpreted as a generalization of the stability problem for the solution of Ito's equation.

Let E^n be an n-dimensional Euclidean space, $E_+ = [0, +\infty)$ and $E_+^{n+1} = E_+ \times E^n$. In the space E^n we consider the ordinary SDE:

$$\xi(t;s,x) = x + \int_s^t b(r,\xi(r))\,dr + \int_s^t c(r,\xi(r))\,dw(r), \tag{1}$$

where $w(t)$, $t \in E_+$ is a d-dimensional Wiener process determined on a complete probability space $\{\Omega, F, P\}$.

Let $F(t,x) = (F_1(t,x),...F_{n-k}(t,x)$ be a smooth vector function. We denote by $M \in E_+^{n+1}$ the smooth manifold given by the equation $F(t,x) = 0$; and by τ_{sx} the first stopping time when the process $(t, \xi(t;s,x))$ attains the manifold *M*. If it cannot attain this manifold, then, by definition, we assume that $\tau_{sx} = +\infty$.

It is obvious that τ_{sx} is a Markov moment with respect to the current of σ-algebra's F_t completed at the initial moment and generated by values of the Wiener process $w(s)$, $t \geq s$.

Theorem 1. Let the functions $b(t,x)$, $c(t,x)$ take values in E^n, $E^d \times E^n$ respectively; and $\xi(t;s,x)$ be the solution of equation [1]. We assume that for some non-negative constants $\lambda_1 + \lambda_2 > 0$:

$$2(F(t,x), \nabla F(t,x) b(t,x) + \frac{1}{2} \Delta F(t,x) c(t,x) c^*(t,x) + |\nabla F(t,x) c(t,x)|^2 + \lambda_1 + \lambda_2 |F(t,x)|^2 \leq 0. \tag{2}$$

Then the manifold *M* is an attractor for the solution $\xi(t;s,x)$.

Proof: Note that in order to prove the theorem it is sufficient to show $\lim_{t \to \infty}$ $F(t,\xi(t,s,x))=0$ with probability 1.

Let θ be an arbitrary Markov stopping time. Since the process $(t\wedge\theta,\ \xi(t\wedge\theta))$, is continuous with probability 1 it has a separable modification and the separable set $\Lambda=\{t_1,t_2,...\}$ can be chosen as independent of θ. By definition the probability of hit for the separable process into the closed set *M* when $t \in [s, +\infty)$ is the same as the one when $t \in \Lambda$. Set B = $\{\omega: \tau_{sx} = +\infty\}$. Note that:

$$B=\bigcap_{k=1}^{\infty}\{\omega:\tau_{sx}>N\}=\bigcap_{k=1}^{\infty}\bigcap_{i=1}^{\infty}\left\{\omega:\left(t_i,\xi(t_i)\right)\in\frac{E_+^{n+1}}{[\mu]_{\frac{1}{k}}}\right\};$$

where $[\mu]_{\frac{1}{k}}=\left\{(t,x):|(t,x)-\mu|<\frac{1}{k}\right\}.$

Hence, it implies in particular that the set B is F- *measurable.*

Using Ito's formula we obtain the estimate:

$$0\le E\left|F(t\wedge\tau,\xi(t\wedge\tau)\right|^2$$

$$=|F(s,x)|^2+E\int_t^{t\wedge t}\{2(F(r,\xi(r)),\nabla F(r,\xi(r))b(r,\xi(r))$$

$$+\Delta F(r,\xi(r))c(r,\xi(r))+\left|\nabla F(r,\xi(r)c(r,\xi(r))\right|^2\}dr$$

$$\le|F(s,x)|^2-\lambda_1E(t\wedge\tau-s)-\lambda_2E\int_s^{t\wedge\tau}\left|F(r,\xi(r))\right|^2dr \qquad (3)$$

We consider the cases when $\lambda_1>0$, $\lambda_2=0$ and $\lambda_1=0$, $\lambda_2>0$ separately. In the first case it follows from (3) that:

$$0\le E\left|F(t,\xi(t))\right|^2\chi(B)\le E\left|F(t\wedge\tau,\xi(t\wedge\tau))\right|^2\le|F(s,x)|^2-\lambda_1(t-x)P(B)$$

If P(B) > 0, then for some t the right-hand side of the inequality above becomes negative. Therefore, P(B)=0 and $\tau_{sx}<\infty$ with probability 1.

Let $\lambda_1 = 0$, $\lambda_2 > 0$. Hence, $E\int_s^{t\wedge\tau} \left|F(r,\xi(r))\right|^2 dr < \lambda_2^{-1}\left|F(s,x)\right|^2$.

Passing in this inequality to the limit when $t\to\infty$ analogously only as before we obtain:

$$E\chi(B)\int_s^{\infty} \left|F(\tau,\xi(\tau))\right|^2 d\tau \le \lambda_2^{-1}\left|F(s,x)\right|^2.$$

If P(B)>0, then almost for every $\omega\in B$, $\int_s^{\infty} \left|F(r,\xi(r))\right|^2 dr < \infty$.

Consequently, $\lim_{t\to\infty} F(t,\xi(t;s,x)) = 0$. Thus, either the process $(t,\xi(t))$ attains *M* during the finite time with probability 1, or this process is asymptotically attracted by the manifold *M*. It follows from (3) that if $(s,x)\in M$ then the process $(t,\xi(t))$ remains there also in the subsequent moments of time.

It implies that in the case when $(s,x)\in M$ the process $(t,\xi(t))$ once having hit into *M* remains there henceforth. In what follows we specify conditions when the solution of equation (1) belongs to the smooth manifold *M*. (Gikhman, Klychkova Ukr. Math. Journ. 1995 v. 47, N2, 174-179).

In addition to the preceding we assume that the coefficients in equation (1) are continuously differentiable on x.

Suppose that $M\in E_+^{n+1}$ is given by the equations

$$F_i(t,x_1,\ldots,x_n) = 0,\ i = 1,2,\ldots,n-k$$

and an arbitrary point (s,y) $\in$ *M* one can find a sphere $S_\delta(s,y) \in E_+^{n+1}$ having radius δ>0 independent of (s,y). We assume that the domain ν_{sy} = *M* $\cap$ $S_\delta(s,y)$ is given by the equations:

$x_o = t$

$x_{jl} = x_{jl,}$ l = 1,2,...k, jl = 1,2,...n

$$x_{jq} = f_{sy}^{(q)}(t,x_{j1},...,x_{jk}),\ q = k+1,...,n,\ j_q = 1,2,...,n.$$

Note that for this purpose it suffices for the matrix

$$\left(\frac{\partial F_i}{\partial x_j}\right),\ l = 1,2\ ...,\ n\text{-}k;\ j = 1,2,...,n$$

to be of the rank n-k. In this case the manifold has dimension k+1. In order to simplify our writing we denote the local coordinates in each neighborhood ν_{sy} by $x_l = x_{jl}$, l=0,1,2,...,k.

Assuming that:

$\varphi(t_1x_1,...,x_k)=$ (4)

$(t_1x_1,...,x_k,f_{sy}^{(k+1)}(t,x_1,...,x_k),...f_{sy}^{(n)}(t,x_1,...,x_k)).$

Theorem 2: Assume that for any arbitrary point (s,y)+*M*, the following conditions are fulfilled:

*) Vector $\vec{e}_0 + \overrightarrow{(b+ch)}(s,y)$, $h \in E^d$ belongs to the tangent manifold $T_{sy}M$.

**) $\sum_{m=1}^{d}\sum_{i,j=1}^{k}\frac{\partial^2 f^{(l)}}{\partial x_i \partial x_j}(s, y_1,...,y_k)\left(c_{im}\ o\varphi\right)\left(c_{jm}\ o\varphi\right)\left(s_1 y_1,...,y_k\right) = 0$,l=k+1,...,n.

Then almost all trajectories process

$$(t,\xi(t;s,y)),\ (s,y)\in M, t > s \text{ belong } M.$$

Proof: In the local map (ν_{sy},φ) we define superposition of functions

$$B(t,x,...,x_k) = (b\circ\varphi)(t,x_1,...,x_k) = b(\varphi(t,x,...,x_k))$$

$$C(t,x,...,x_k) = (c\circ\varphi)(t,x,,...,x_k) = c(\varphi(t,x,...,x_k))$$

For each $t\in[s,s+\delta)$ there exists a smooth extension of the functions B(•), C(•) to all E^k. Therefore there exists a unique solution of equation [1] with coefficients B and C. Let τ_{sx} be the moment of the first exit of the process $(r,\xi(r;s,x))$ out of $S_\delta(s,x)$. If for some ω this process does not attain the exterior of $S_\delta(s,x)$ then $\tau_{sx}= t$. We denote by $\vec{e}_0,...,\vec{e}_\nu$ the initial basis in E_+^{n+1}.

Set

$$\vec{N}_\ell(s,y) = \vec{e}_0\frac{\partial f_{sy}^{(\ell)}(s,y,...,y_k)}{\partial t} + \sum_{i=1}^{k}\vec{e}_i\frac{\partial f_{sy}^{(\ell)}(s,y,...,y_k)}{\partial x_i} - \vec{e}_\ell,\ \ \ell = k+1,...,n$$

Note that:

$$(N_{\ell'},N_{\ell''}) \neq 0,\ \ \ell' \neq \ell''$$

$$(\vec{N}_\ell,\nabla\varphi_i) = 0,\ \ i = 1,2,...,k$$

that is, $\vec{N}_\ell(s,y),\ \ \ell = k+1,...,n$ form nonorthogonal basis in orthogonal complement to $T_{sy}M$. Using Ito's formula and conditions of the theorem 2 we can prove that

$$\xi_1(t\wedge\tau;s,y) = f_{sy}^{(\ell)}(t\wedge\tau,\xi_1(t\wedge\tau)...,\xi_k(t\wedge\tau)),\qquad \ell = k+1,...,n$$

So, by definition

$$(t\wedge\tau,\xi(t\wedge\tau;s,y))\in M$$

To complete the proof of the theorem we have to repeat the technique of the solution construction starting from the random point $(t\wedge\tau,\xi\ (t\wedge\tau))$.