CONIC SECTIONS

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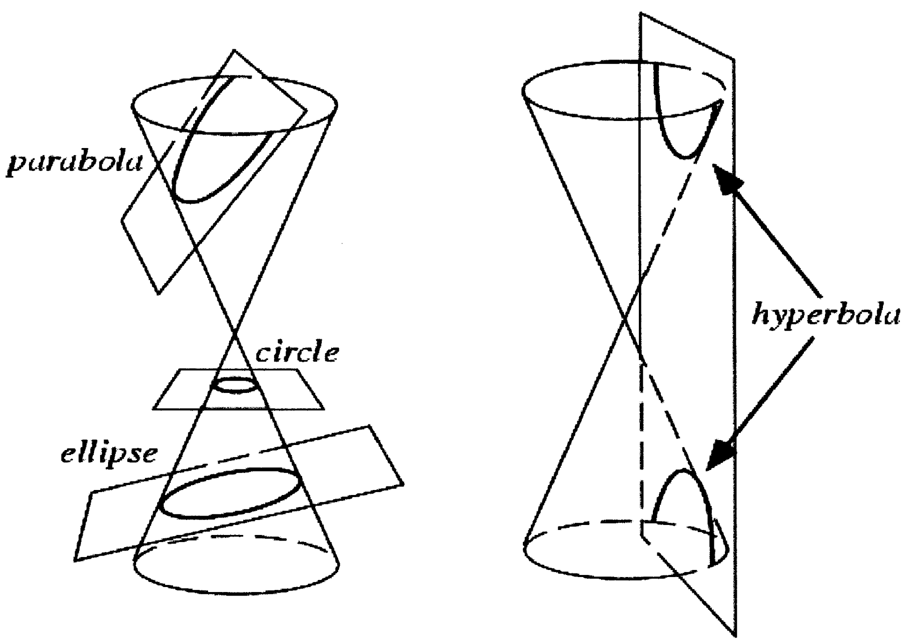
HYPATIA – 370?-415 CE

It was Apollonius who was the first to note that the conic sections could be constructed apart from algebraic equations by cutting the right--circular cone with a plane. As a matter of fact, Apollonius did not note the connection of the conics to their algebraic equations. These equations did not enter the mathematical picture for hundreds of years.

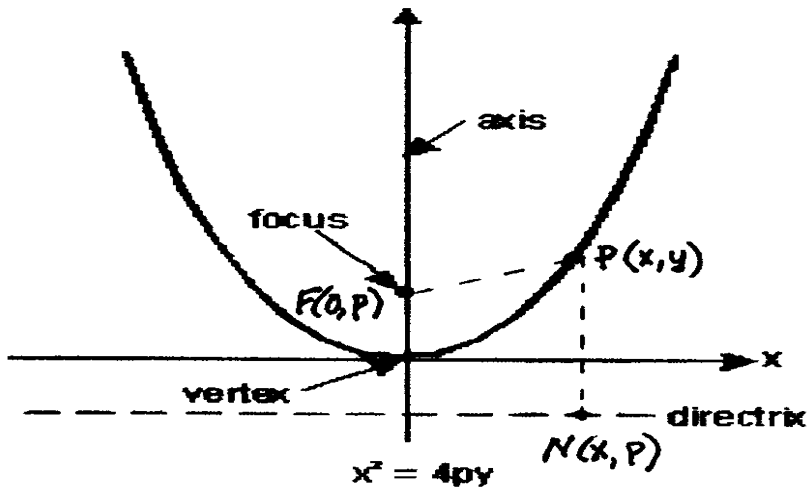
Hypatia was known primarily for her work on the ideas of conic sections introduced by Apollonius. She edited the work *On the Conics of Apollonius,* which divided cones into different parts by a plane. This concept developed the ideas of **hyperbolas, parabolas, and ellipses**. With Hypatia's work on this important book, she made the concepts easier to understand, thus making the work survive through many centuries. Hypatia was the first woman to have such a profound impact on the survival of early thought in mathematics.

The four conic sections are: **parabolas, circles, hyperbolas** and **ellipses**.



THE PARABOLA

A *parabola* is the set of all points (x,y) that are equidistant from a fixed line (directrix) and a fixed point (focus) not on the line. In the following diagrams, the line segment FP is equal to the line segment PN.

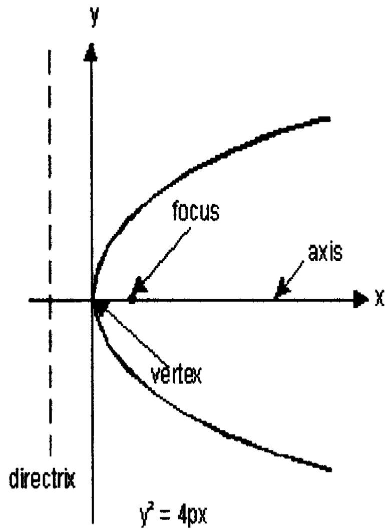


PF = PN

PF	=	PN
$\sqrt{(x-0)^2+(y-p)^2}$	=	$\sqrt{(x-x)^2+(y+p)^2}$
$x^2 + (y - p)^2$	=	$(y + p)^2$
$x^2 + y^2 - 2yp + p^2$	=	$y^2 + 2yp + p^2$
x^2	=	$4py$

$$y = \frac{x^2}{4p} = \frac{1}{4p}x^2$$

$$\boxed{y = \frac{1}{4p}x^2}$$



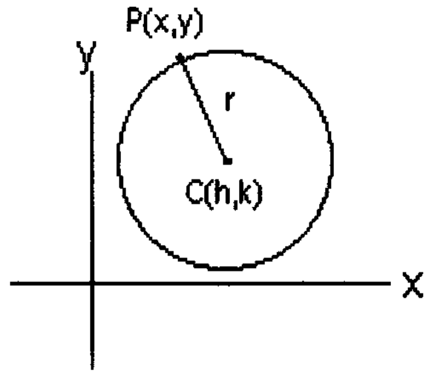
Note: If $y = x^2$ then the inverse is $x = y^2$. Check both of these on your graphing calculator.

Now, the inverse of $y = \frac{1}{4p}x^2$ is $x = \frac{1}{4p}y^2$ and solving for y^2 we have: $y^2 = 4px$.

Remember, when you translated axis for $y = x^2$, you have $y=(x-h)^2 + k$

THE CIRCLE

A *Circle* is the set of points all equidistant from a given point, the center.



$$d(\overline{PC}) = \sqrt{(x-h)^2 + (y-k)^2}$$

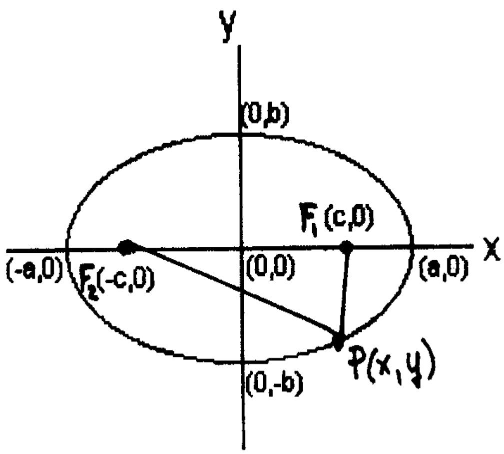
$$r^2 = (x - h)^2 + (y - k)^2$$

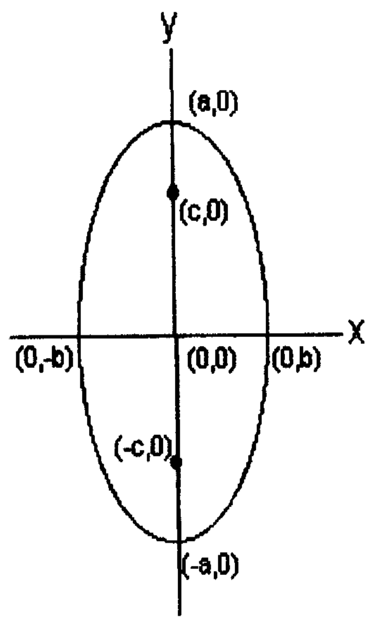
or $(x - h)^2 + (y - k)^2 = c^2$ **(c is constant).**

THE ELLIPSE

***Ellipse* – If F_1 (c, o) and F_2 (-c, o) are fixed points in the plane, and *a* is constant, 0 < c < a, then the geometric definition of an ellipse states that a ellipse is the set of all points P(x,y) in the plane such that:**

$$|PF_1 + PF_2| = 2a$$





$$|PF_1 \quad + \quad PF_2| \quad = \quad 2a$$

$$|PF_1 \quad + \quad PF_2| \quad = \quad 2a$$

$$\sqrt{(x-c)^2+(y-0)^2} \quad + \quad \sqrt{(x+c)^2+(y-0)^2} \quad = \quad 2a$$

$$\sqrt{(x-c)^2+y^2} \quad + \quad \sqrt{(x+c)^2+y^2} \quad = \quad 2a$$

$$\left(\sqrt{(x+c)^2+y^2}\right)^2 \quad = \quad \left(2a-\sqrt{(x-c)^2+y^2}\right)^2$$

$$(x+c)^2+y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + [(x-c)^2 + y^2]$$

$$x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + x^2 - 2cx + c^2 + y^2$$

$$4cx = 4a^2 - 4a\sqrt{(x-c)^2+y^2}$$

$$cx = a^2 - a\sqrt{(x-c)^2+y^2}$$

$$cx - a^2 = -a\sqrt{(x-c)^2+y^2}$$

$$\left(cx-a^2\right)^2 = \left(-a\sqrt{(x-c)^2+y^2}\right)^2$$

$$a^4 - 2acx + c^2x^2 = a^2(x-c)^2 + y^2$$

$$a^4 - 2acx + c^2x^2 = a^2(x^2 - 2cx + c^2) + y^2$$

$$a^4 - 2acx + c^2x^2 = a^2x^2 - 2ca^2x + a^2c^2 + a^2y^2$$

$$a^4 + c^2x^2 = a^2x^2 + a^2c^2 + a^2y^2$$

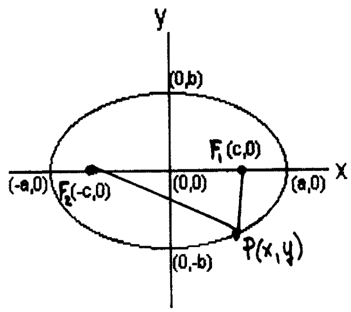
$$a^4 - a^2c^2 = a^2x^2 - c^2x^2 + a^2y^2$$

$$a^2(a^2 - c^2) = (a^2 - c^2)x^2 + a^2y^2$$

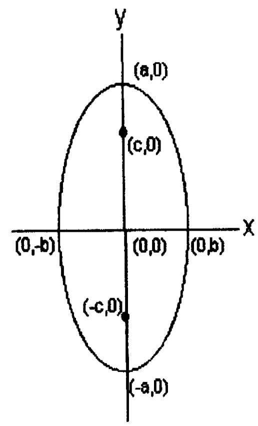
Substitute: $b^2 = a^2 - c^2$ $\quad a^2b^2 = b^2x^2 + a^2y^2$

Divide by a^2b^2 $\quad 1 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

So, $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad$ **where** $\quad b^2 = a^2 - c^2$



$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

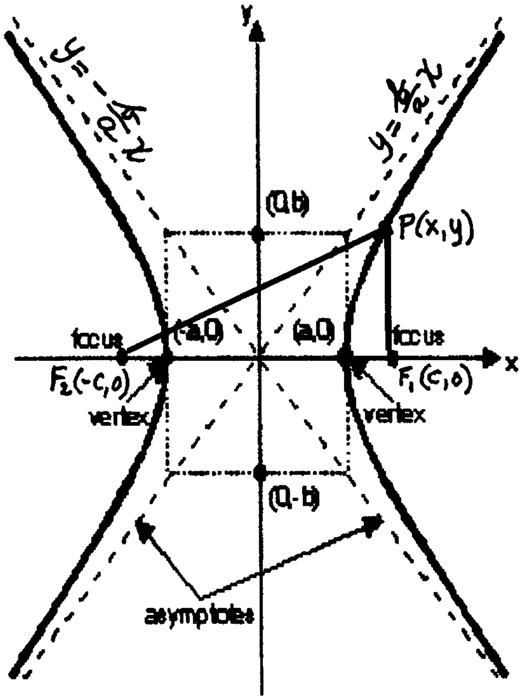


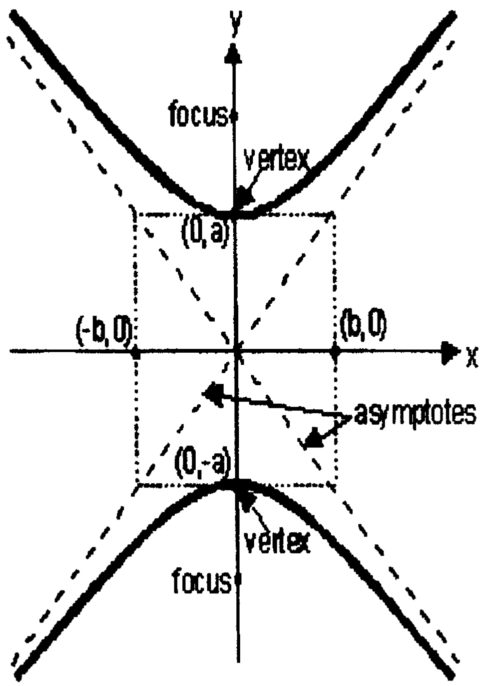
$$\frac{x^2}{b^2}+\frac{y^2}{a^2} = 1$$

THE HYPERBOLA

***Hyperbola* – Suppose that F_1 (c, 0) and F_2 (-c, 0) are fixed points in the plane, and that *__a__* is constant, a < c. The geometric definition of a hyperbola states that a hyperbola is the set of all points P(x,y) in the plane such that:**

$$|PF_1 - PF_2| = 2a$$





$$|PF_1 \quad - \quad PF_2| \quad = \quad 2a$$

$$\sqrt{(x-c)^2 + y^2} \quad - \quad \sqrt{(x+c)^2 + y^2} \quad = \quad \pm 2a$$

We do the same thing with the above equation as we did with the ellipse, except we let $b^2 = c^2 - a^2$ and the result is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Solve for y:

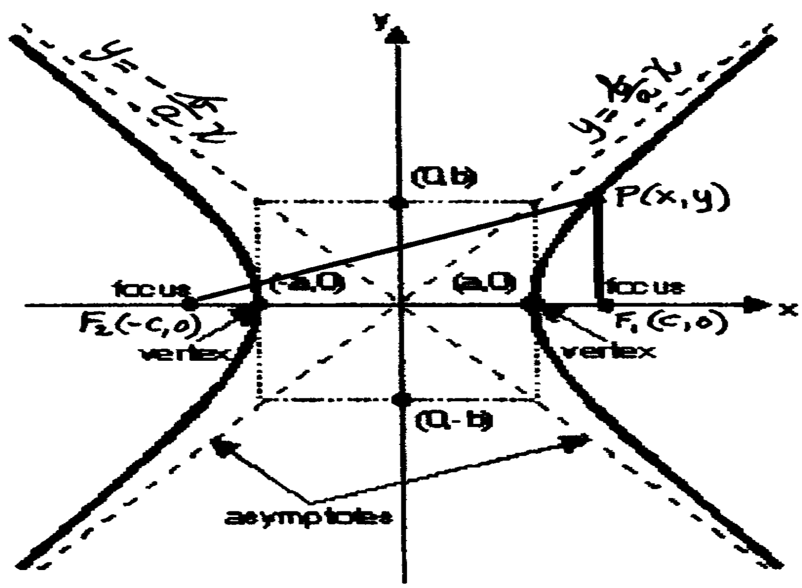
$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$

$$y^2 = b^2\left(\frac{x^2 - a^2}{a^2}\right)$$

$$y^2 = \frac{b^2}{a^2}\left(x^2 - a^2\right)$$

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$$

Where $x \to \infty$ $\quad y \approx \pm\frac{b}{a}x$ **(Asymptotes)**



If the above graph is of: $\frac{y^2}{9} - \frac{x^2}{36} = 1$**, find its foci.**

When $x = 0$, $y^2 = 9$ implies $y = \pm 3$

To find the foci: $c^2 = a^2 + b^2 = 9 + 36 = 45$

Thus, $\quad c = \pm\sqrt{45} = \pm 3\sqrt{5}$

So, the foci are: $(0, \pm 3\sqrt{5})$