HISTORY OF ORDINARY DIFFERENTIAL EQUATIONS

THE FIRST HUNDRED YEARS

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ABSTRACT

The author gives a brief description of the development of general methods of integrating ordinary differential equations from its beginning in 1675 until 1775 when the search for such methods ended. The focus of the paper is the historical roots of nine mathematical problems that led to the independent discipline now called Ordinary Differential Equations.

MATHEMATICAL ORIGINS OF ORDINARY DIFFERENTIAL EQUATIONS: THE FIRST HUNDRED YEARS

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The attempt to solve physical problems led gradually to mathematical models involving an equation in which a function and its derivatives play important roles. However, the theoretical development of this new branch of mathematics - Ordinary Differential Equations - has its origins rooted in a small number of mathematical problems. These problems and their solutions led to an independent discipline with the solution of such equations an end in itself.

According to some historians of mathematics, the study of differential equations began in 1675, when Gottfried Wilhelm von Leibniz (1646-1716) wrote the equation:

$$\int x\,dx = (1/2)x^2. \quad [16].$$

The search for general methods of integrating differential equations began when Isaac Newton (1642-1727) classified first order differential equations into three classes: **[24].**

$$(1)\; dy/dx = f(x)$$

$$(2)\; dy/dx = f(x,y)$$

$$(3)\; x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$$

The first two classes contain only ordinary derivatives of one or more dependent variables, with respect to a single independent variable, and are known today as ordinary differential equations. The third class involved the partial derivatives of one dependent variable and today are called partial differential equations.

Newton would express the right side of the equation in powers of the dependent variables and assumed as a solution an infinite series. The coefficients of the infinite series were then determined **[23].**

Even though Newton noted that the constant coefficient could be chosen in an arbitrary manner and concluded that the equation possessed an infinite number of particular solutions, it wasn't until the middle of the 18th century that the full significance of this fact, i.e., that the general solution of a first order equation depends upon an arbitrary constant, was realized.

Only in special cases can a particular differential equation be integratable in a finite form, i.e., be finitely expressed in terms of known functions. In the general case one must depend upon solutions expressed in infinite series in which the coefficients are determined by recurrence-formulae.

In 1682, Leibniz became a collaborator on the new Leipzig periodical, *Acta Eruditorum*, in which he published his epoch-making six page paper on the differential calculus in 1684 **[11],** followed two years later (1686) by a paper containing the rudiments of the integral calculus. **[19].**

James Bernoulli (1654-1705] wrote Leibniz in 1687 requesting initiation into the mysteries of the new analysis. Because Leibniz was travelling abroad, Bernoulli's letter remained unanswered until 1690.

In the meantime James and his brother John Bernoulli (1667-1748) unravelled the mysteries without assistance. Their success initiated an extensive correspondence with

Leibniz. In these letters are contained many innovations and anticipations of since prominent methods **[25].** In 1692 James Bernoulli made known the method of integrating the homogeneous differential equation of the first order, and not long afterwards reduced to quadratures the problem of integrating a linear equation of the first order **[14].**

The original discoveries of practically all known elementary methods of solving differential equations of the first-order took place during the Bernoulli dynasty.

The Bernoullis' were a Swiss family of scholars whose contributions to differential equations spanned the late seventeenth and the eighteenth century. Nikolaus Bernoulli I (1623-1708) was the progenitor of this celebrated family of mathematicians. James I, John I, and Daniel I are the best known members of the Bernoulli family who made many contributions to this new field of differential equations.

In May, 1690, James Bernoulli published in Acta eruditorum his solution of the problem of the isochrone **[13,19].** The problem of the isochrone involved the isochronous curve, a curve along which a body will fall with uniform vertical velocity. This problem led him to a differential equation expressing the equality of two differentials.

From this James Bernoulli concluded that the integrals of the two members of the equation were equal and used the word integral for the first time on record in Acta Eruditorum in 1696 **[3].** Thus, the second of the two major divisions of calculus, called *calculus summatorius* at the time, was changed to *calculus integralis*, or, as we know it today, integral calculus.

From the problem of the isochrone came the idea of obtaining the equation of a curve which has a kinematical or a dynamical definition by expressing the mode of its description in the guise of a differential equation, and then integrating this equation under certain initial conditions. The *logarithmic spiral* is an example of such a curve.

In 1691 the inverse problem of tangents led Leibniz to the implicit discovery of the method of separation of variables **[15].**

However, it was John Bernoulli, in a letter to Leibniz, dated May 9, 1694, that gave us the explicit process and the term, *seperatio indeterminatarum* or separation of variables **[6].**

But even then, in one, but important, case of:

$$x\,dy - y\,dx = 0$$

this process failed, because it led to 1/y dy = 1/x dx and 1/x dx had not yet been integrated.

In the same year, however, Leibniz solved the *problem of the quadrature of the hyperbola* - a process of finding a square equal in area to the area under the curve on a given interval.

John Napier's work on logarithms, published eighty years before, made possible Leibniz's rendition of the integration of the differential 1/x dx as log x **[14].**

In 1696, John Bernoulli, a student, rival, and equal of his older brother James, gave a main impetus to the study of differential equations through posing his famous brachistochrone problem of finding the equation of the path down which a particle will fall from one point to another in the shortest time **[7].**

The equation:

$$dy/dx + P(x)y = f(x)y^n$$

known today as the Bernoulli equation, was proposed for solution by James Bernoulli in December, 1695 **[3].**

The following year Leibniz solved the equation by making substitutions and simplifying to a linear equation, similar to the method employed today **[18].**

Then in 1698, John Bernoulli solved the *problem of determining the orthogonal trajectories of a one-parameter family of curves* - finding a curve which cuts all the curves of a family of curves at right angles. With this solution was laid to rest the problem of oblique trajectories as well.

Virtually all known elementary methods of solving first order differential equations had been found by the end of the seventeenth century.

In the early years of the eighteenth century a number of problems led to differential equations of the second and third order. In 1701 James Bernoulli published the solution to the *isoperimetric problem* - a problem in which it is required to make one integral a maximum or minimum while keeping constant the integral of a second given function - thus resulting in a differential equation of the third order. **[4,18].** In a letter written to Leibniz, May 20, 1716, John Bernoulli discussed the equation:

$$d^2y/dx^2 = 2y/x^2$$

where the general solution when written in the form

$$y = x^2/a + b^2/3x$$

involves three cases: When b approaches zero the curves are parabolas; when a approaches infinity, they are hyperbolas; otherwise, they are of the third order **[7].**

Jacopo Riccati's (1676-1754) discussion of special cases of curves whose radii of curvature were dependent solely upon the corresponding ordinates resulted in his name being associated with the equation

$$\mathbf{y' = P(x) + Q(x)y + R(x)y^2.}$$

Riccati's discussion offered no solutions of his own. It was Daniel Bernoulli who successfully treated the equation commonly known as the *Riccati Equation*. **[2].**

In general this equation cannot be solved by elementary methods. However, if a particular solution $y_1(x)$ is known, then the general solution has the form:

$$\mathbf{y(x) = y_1(x) + z(x)}$$

where z(x) is the general solution of the Bernoulli equation:

$$\mathbf{z' - (q+2ry_1)z = rz^2.}$$

By 1724 Daniel Bernoulli had found the necessary and sufficient conditions for integrating in a finite form the equation:

$$\mathbf{y' + ay^2 = bx^m.}$$

Leonhard Euler (1707-1783) provided the next significant development when he posed and solved the problem of *reducing a particular class of second order differential equations to that of first order.* His process of finding a second solution from a known solution consists both of reducing a second order equation to a first order equation and of *finding an integrating factor*. Additionally, Euler confirmed that the ratio of two different integrating factors of a first-order differential equation is a solution of the equation **[12].**

Euler began his treatment of the homogeneous linear differential equation with constant coefficients in a letter he wrote to John Bernoulli on September 15, 1739, published in *Miscellanea Berolinensia*, 1743. Within a year Euler had completed this treatment by successfully dealing with repeated quadratic factors and turned his attention to the non-homogeneous linear equation **[12].**

The method of successive reduction of the order of the equation with the aid of integrating factors led first to equations integrable in finite form. Euler first reduced these equations step by step and then integrated. For those equations which were not integrable in a finite form, Euler used the method of integrating by series.

Alexis Claude Clairaut (1713-1765) applied the process of differentiation to the equation:

$$\mathbf{y = x\,\frac{dy}{dx} + f\,\frac{dy}{dx}}$$

which is now known as *Clairaut's equation* and in 1734 published his research on this class of equations **[9].** Clairaut was among the first to solve the *problem of singular solutions* - finding an equation of an envelope of the family of curves represented by the general solution **[20].**

Joseph Louis Lagrange (1736-1813), while working on the *problem of determining an integrating factor for the general linear equation*, formalized the concept of the *adjoint equation*.. Lagrange not only determined an integrating factor for the general linear equation, but furnished proof of the general solution of a homogeneous linear equation of order ***n*** **[20].** In addition Lagrange discovered the method of variation of parameters. **[21].**

Building on Lagrange's work, Jean Le Rond d'Alembert (1717-1783), found the *conditions under which the order of a linear differential equation could be lowered* **[10].** By deriving a method of dealing with the exceptional cases, d'Alembert solved the *problem of linear equations with constant coefficients*, and initiated the study of *linear differential systems* **[10].** In a treatise written in 1747, devoted to vibrating strings, d'Alembert was led to partial differential equations where he did his main work in the field.

The period of initial discovery of general methods of integrating ordinary differential equations ended by 1775, a hundred years after Leibniz inaugurated the integral sign. For many problems the formal methods were not sufficient. Solutions with special properties were required, and thus, criteria guaranteeing the existence of such solutions became increasingly important. Boundary value problems led to ordinary differential equations, such as Bessel's equation, that prompted the study of Laguerre, Legendre, and Hermite polynomials. The study of these and other functions that are solutions of equations of hypergeometric type led in turn to modern numerical methods.

Thus, by 1775, as more and more attention was given to analytical methods and problems of existence, the search for general methods of integrating ordinary differential equations ended.

SUMMARY
(1675-1775)

DATE	PROBLEM	DESCRIPTION	MATHEMATICIAN
1690	Problem of the Isochrone	Finding a curve along which a body will fall with uniform vertical velocity	James Bernoulli
1694	Quadrature of the Hyperbola	Process of finding a square equal to the area under the curve on a given interval	G.W. Leibniz
1696	Brachistochrone Problem	Finding the equation of the path down which a particle will fall from one point to another in the shortest time	John Bernoulli
1698	Problem of Orthogonal Trajectories	Finding a curve which cuts all the curves of a family of curves at right angles	John Bernoulli
1701	Isoperimetric Problem	A problem in which it is required to make one integral a maximum or minimum while keeping constant the integral of a second given function	Daniel Bernoulli
1728	Problem of Reducing 2nd Order Equations to 1st Order	Finding an integrating factor	Leonhard Euler
1734	Problem of Singular Solutions	Finding an equation of an envelope of the family of curves represented by the general solution	Alexis Clairaut
1743	Problem of determining integrating factor for the general linear equation	Concept of the adjoint of a differential equation	Joseph Lagrange
1762	Problem of Linear Equation with Constant Coefficients	Conditions under which the order of a linear differential equation could be lowered	Jean d'Alembert

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