

THE IMPORTANCE OF BOTH RIGOR  
AND  
INTUITION IN MATHEMATICS

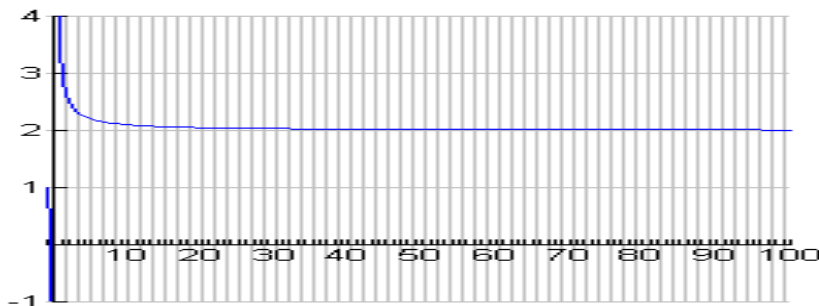
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The purpose of this article is to point out the importance of both rigor and intuition in mathematics.

Consider the equation  $y = (2x + 1)/x$ . Let us replace  $x$  with the first positive integer, 1, and find the corresponding value for  $y$ :  $y = [2(1) + 1]/1 = 3$ . What value would we obtain if we continued to replace  $x$  with each succeeding positive integer, 2,3,4,... ?

x	1	2	3	4	5	...	100	...	1000	...	$10^5$
y	3	2.5	2.3	2.25	2.2		2.01		2.001		2.00001

If we graph this equation we obtain:



It appears that the larger we make  $x$ , the closer the value of  $y$  will approach 2.

When we say that  $y$  “appears” to approach 2, we mean it is intuitively obvious that  $y$  will get closer and closer to 2 as  $x$  becomes larger and larger.

“Intuitively obvious” means convincing in the absence of proof. We are convinced that the sequence,  $(2n+1)/n$  from  $n=1$  to infinity, equals 2 because of the way it fits in with everything else we know.

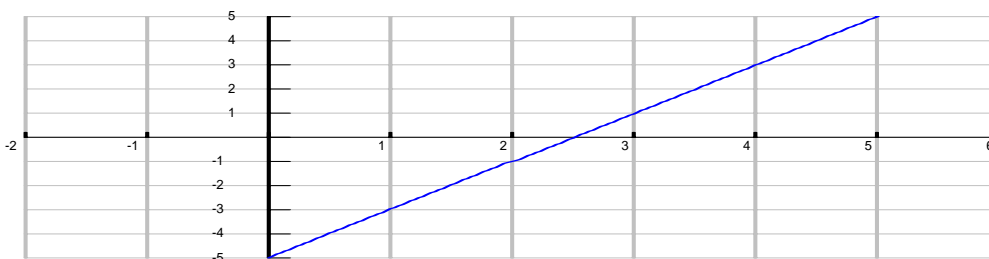
We have been reasoning intuitively. If we justified our result deductively, that is, reasoning from a known first step to our conclusion, defining each step, we would “rigorously prove” our result.

Let us look at another example:  $f(x) = 2x - 5$ .

What value does  $f(x)$  approach as  $x$  approaches 3?

x	0	1	2	2.5	2.8	2.9	...	2.99	...	2.999
y	-5	-3	-1	0	0.6	0.8		0.98		0.998

Again, it is intuitively obvious that  $f(x)$  approaches 1 as  $x$  approaches 3. At least as  $x$  approaches 3 from the left.



We write this as  $\lim_{x \rightarrow 3^-} f(x) = 1$  and read it as “ $f(x)$  becomes arbitrarily close to 1 as  $x$  approaches 3 from the left side.”

What number does  $f(x)$  become arbitrarily close to as  $x$  approaches 3 from the right side?

$x$	2.999	3	3.001	3.01	4
$y$	0.998	?	1.002	1.02	3

Thus, it appears that  $f(x)$  approaches 1 as  $x$  approaches 3 from the right.

If  $f(x)$  becomes arbitrarily close to a single number  $L$  as  $x$  approaches  $c$  from either side, we write:

$$\lim_{x \rightarrow c} f(x) = L$$

and say, the limit of  $f(x)$ , as  $x$  approaches  $c$ , is  $L$ .

**Rigorous development of the limit concept.**

Our discussion of limits thus far has been strictly at the intuitive level. We have used expressions such as these:

*“For  $x$  near  $c$ ,  $f(x)$  is near  $L$ .”*  
*“ $F(x)$  approaches  $L$  as  $x$  approaches  $c$ .”*

Intuitively, we said that the limit statement:

$$\lim_{x \rightarrow c} f(x) = L$$

means “ $f(x)$  approaches  $L$  as  $x$  approaches  $c$ .”

This statement can be interpreted to mean that the difference  $|f(x) - L|$  is arbitrarily small provided the difference  $|x-c|$  is sufficiently small.

In other words, for each small epsilon ( $\epsilon$ )  $> 0$  there must be a small delta ( $\delta$ ) such that:

$$|f(x) - L| < \epsilon \text{ whenever } |x-c| < \delta$$

Recall that for limits at  $c$  we are not interested in the value of  $f$  at  $x = c$ . Thus, we will further require  $0 < |x-c|$ , which implies that  $x \neq c$ . In the following definition of the limit of  $f$  at  $c$ , we assume that  $f$  is defined on an open interval containing  $c$ , except possibly at  $c$  itself.

**Definition of Limit:** The statement,

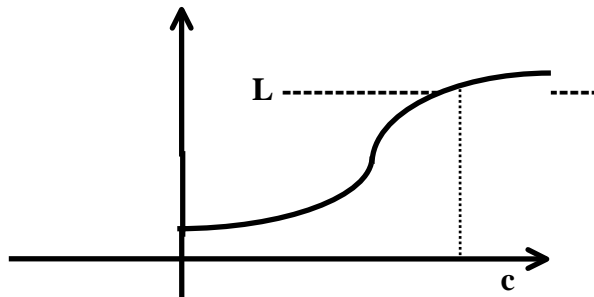
$$\lim_{x \rightarrow c} f(x) = L$$

means that for each  $\epsilon > 0$  there exists a  $\delta > 0$  such that,

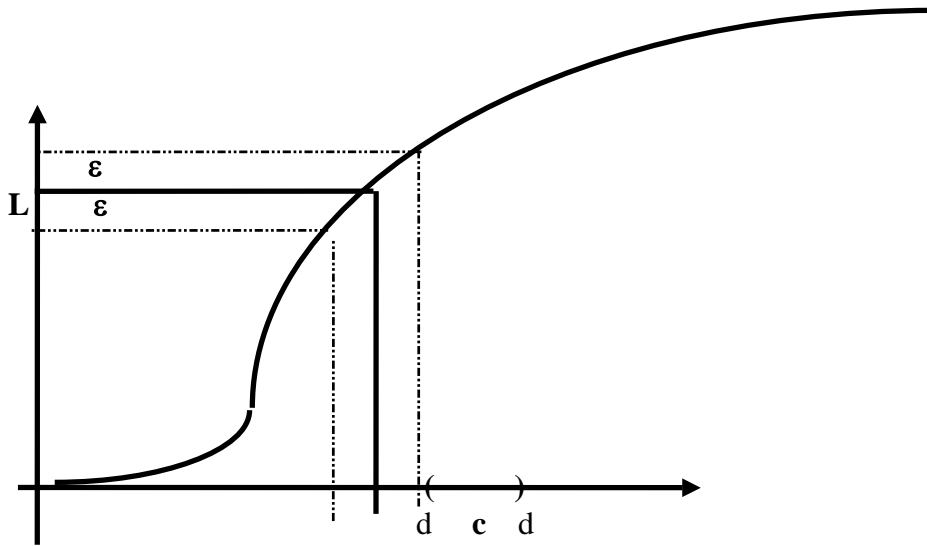
$$|f(x) - L| < \epsilon \text{ whenever } 0 < |x-c| < \delta.$$

Note that the definition has an order to it: “for each  $\epsilon > 0$  there exists a  $\delta > 0$ .” This means that if we choose a particular value for  $\epsilon$ , then for this choice  $\epsilon$  there exists a  $\delta$  that works. We do not require any specific  $\delta$  to work for more than one choice of  $\epsilon$ . Furthermore, the number  $\delta$  is not unique, for if a specific  $\delta$  works, then any smaller positive number will also work.

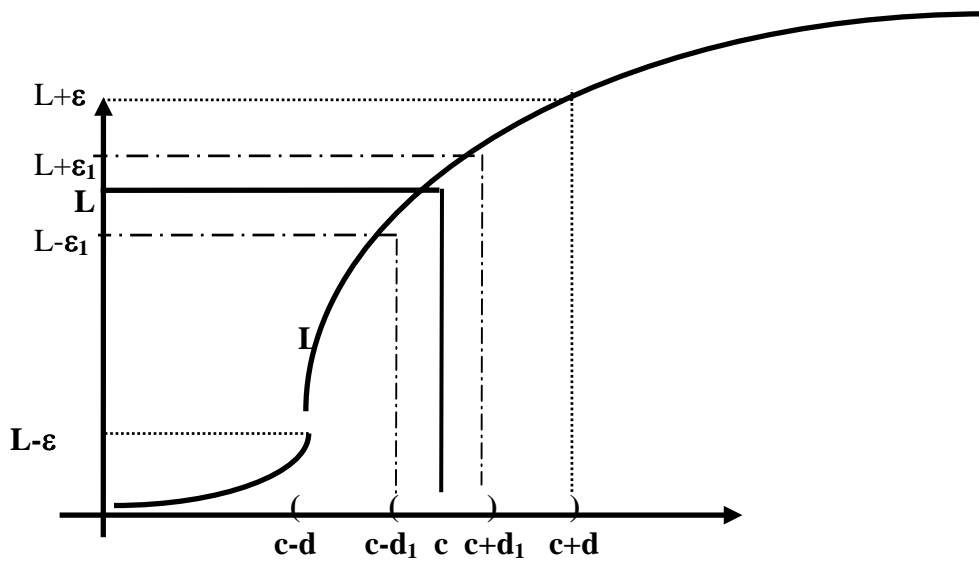
Suppose  $\lim_{x \rightarrow c} f(x) = L$



then for any choice of  $\epsilon > 0$  there exists a  $\delta > 0$ .



Such that  $L-\epsilon < f(x) < L+\epsilon$  whenever  $c-d < x < c+d$ .



If we pick another smaller  $\epsilon_1$ , it can require a new choice  $\delta_1 < \delta$ .

The examples that follow illustrate some ways to determine the connection between  $\epsilon$  and  $\delta$ , for a particular limit.

We will write the definition:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } |f(x) - L| < \varepsilon \text{ whenever } 0 < |x - c| < \delta$$

“For every epsilon greater than zero, there exist a delta greater than zero such that the absolute value of  $f(x)$  minus  $L$  is less than epsilon whenever the absolute value of  $x$  minus  $c$  is greater than zero but less than delta.”

**Example:**

1.) Prove  $\lim_{x \rightarrow 3} (2x - 5) = 1$ .

To show:  $|(2x-5) - 1| < \varepsilon$  whenever  $0 < |x-3| < \delta$ .

Since we know our choice for  $\delta$  is dependent on our choice of  $\varepsilon$ , we must establish a connection between  $\delta$  and  $\varepsilon$ .

This requires that we establish a connection between

$$|(2x-5) - 1| \text{ and } |x-3|$$

$$|(2x-5) - 1| = |2x-6| = |2(x-3)| = 2|x-3|$$

Thus, the statement:

$$|(2x-5) - 1| = 2|x-3| < \varepsilon$$

requires

$$|x-3| < \frac{\varepsilon}{2}$$

which means we can choose  $\delta \leq \frac{\varepsilon}{2}$ .

This choice works because whenever:

$$0 < |x-3| < \delta \leq \frac{\varepsilon}{2}$$

$$\text{then } |(2x-5) - 1| = 2|x-3| < 2\left(\frac{\varepsilon}{2}\right) = \varepsilon.$$

The above “rigorous proof” might very well be less convincing to many students than our intuitive argument. Thus, it is understandable that one might ask why we need rigorous proofs at all. As a matter of fact, even the above rigorous proof was not possible without first intuitively knowing the limit!

However, rigor is necessary because intuition can be deceptive. For example, for a hundred years, it was intuitively obvious to mathematicians, including Cauchy, the founder of rigor, that a continuous function must have a derivative. Today, we know there are many such continuous functions that do not have a derivative.

Also, rigor is necessary since this is the only way most mathematics can be communicated intelligently.

As we have already seen, intuition is important, because most rigorous proofs are for facts that our intuition first tells us are true.