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<u>A B S T R A C T</u>

A RESTATEMENT OF GOLDBACH'S CONJECTURE

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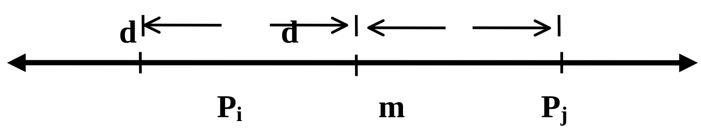
Christian Goldbach conjectured in a letter he wrote Leonard Euler in 1742 that every even integer greater than two can be written as the sum of two primes.

In this paper, the author has re-stated Goldbach's Conjecture in terms of midpoints between primes. Following this restatement it is then shown that there are p(p-1)/2 midpoints between p consecutive prime numbers, and further, that all the midpoints are positive integers. Then the ratio of integral midpoints to the primes is summarized.

Finally, letting $P(X) = 1 - \frac{no.of\ surjective\ functions}{no.of\ all\ functions}$ the limit of P(x) as $x \to \infty$ is found to be 0.

GOLDBACH'S CONJECTURE

Assuming Goldbach's conjecture, that every even integer greater than two can be written as the sum of two primes, is true: If one wishes to find two primes, the sum of which is a given integer, Z, then divide Z by 2, let Z/2 = m, and search for two prime numbers, P_i and P_j which are equal distance from m:

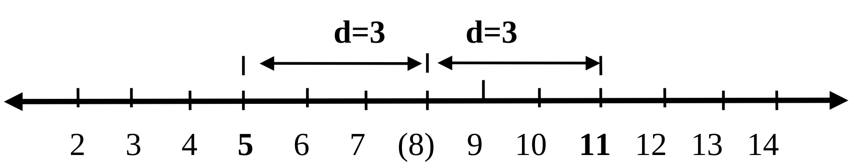


$m - d = P_i$ and $m + d = P_j$

implies **$(m-d) + (m+d) = P_i + P_j$**

implies **$2m = Z = P_i + P_j$.**

For example, if you wish to write 16 as the sum of two primes, then m = 16/2 = 8, and search for two primes which are equal distance from 8:



(8 - 3) = 5 and (8 + 3) = 11

Thus, 2(8) = 16 = 5 + 11.

Since 4 = 2+2 and m = any integer equal to or greater than 3, then every even integer greater than two can be written as the sum of two primes, ***provided that the set of integers equal to or less than n, which are midpoints between primes includes all the integers that are equal to or less than n.***

The following is based on the above re-statement of Goldbach's conjecture, i.e.:

RE-STATEMENT OF GOLDBACH'S CONJECTURE

> **Every even integer greater than two can be written as the sum of two primes, provided that the set of integers which are midpoints between primes,**

$$J(X)=\frac{\frac{x}{\ln x}\left[\frac{x}{\ln x}-1\right]}{2},$$

includes all the integers greater than 2.

J = {3, 4, 5, ...}.

That is, J is a subset of J(x).

1) <u>There exist p(p-1)/2 midpoints between p consecutive prime numbers.</u>

$$\text{Given}[P_1, P_2, P_3, ..., P_{p-1}, P_p]$$

$$\text{Then}\ \frac{P_1+P_2}{2}, \frac{P_1+P_2}{2}, ..., \frac{P_1+P_{p-1}}{2}, \frac{P_1+P_p}{2}\ \text{implies}\ p-1\ \text{midpts;}$$

$$\frac{P_2+P_3}{2}, ..., \frac{P_2+P_{p-1}}{2}, \frac{P_2+P_p}{2}\ \text{implies}\ p-2\ \text{midpts;}$$

$$..., \frac{P_3+P_{p-1}}{2}, \frac{P_3+P_p}{2}\ \text{implies}\ p-3\ \text{midpts;}$$

$$..., ..., ..., ...,$$

$$\frac{P_3+P_p}{2}\ \text{implies}\ p-(p-1)\text{midpts.}$$

$$\text{Now,}(p-1)+(p-2)+(p-3)+...[p-(p-1)]$$

$$=p\bullet p-\frac{p(p+1)}{2}=p^2-\frac{p^2+p}{2}$$

$$=\frac{2p^2-(p^2+p)}{2}=\frac{p(p-1)}{2}.$$

Therefore, if p = the number of consecutive primes, then $\frac{p(p-1)}{2}$ equals the total number of midpoints.

2) <u>Further, all the midpoints are positive integers.</u>

For P > 2, all primes are odd. The sum of two odds is even, and every even integer divided by two is an integer. Thus, we have $\frac{p(p-1)}{2}$ positive integers as midpoints between p primes.

3) <u>The number of primes Π(x), [equal to or less than some sufficiently large number, x] is approximately x/(ln x).</u>

We know this because the ratio of Π(x) to $\frac{x}{\ln x}$ approaches 1, as x approaches infinity, that is:

$$\lim_{x\to\infty}\left(\frac{\Pi(x)}{\frac{x}{\ln x}}\right)$$

$$\lim_{x\to\infty}\left[\frac{\Pi(x)\ln x}{x}\right] \approx 1.$$ (Prime Number Theorem).

4) <u>Summary Table:</u>

Total Integers	Primes	Integral midpoints
n	p	$\frac{p(p-1)}{2}$
23	9	36
100	25	300
1000	168	14028
10000	1229	754606
100000	9592	45998436
1000000	78498	3080928753

We see from the table that within the first one million integers there are 78,498 prime numbers, which have contained between them a total of three billion midpoints. Of course, many of these midpoints represent the same integer. Note, for the first million integers there are 3,000 midpoints per integer, and the ratio increases as the number of integers increases.

(5) **<u>Finally, if it can be shown that each of the integers is in fact represented by at least one of the midpoints, then Goldbach's conjecture is proven.</u>**

If m midpoints are assigned to x integers at random, we find the probability, P(x), that every integer is a midpoint, i.e., $\lim_{x\to\infty} P(x) = 1.$

We have shown:

$$\frac{p(p-1)}{2} = \text{number of midpoints between p consecutive primes.}$$

$$\frac{x}{\ln x} \approx \text{number of primes equal to or less than some large x.}$$

thus, $$\frac{\frac{x}{\ln x}\left[\frac{x}{\ln x}-1\right]}{2} \approx m = \text{integers recurring as midpoints (averages) of}$$

pairs of primes > 2, *counting multiplicities*.

<u>If m midpoints are placed on x integers at random, find the probability, P(x), that no integer is omitted.</u>

The number of ways to place m markers on the first x integers is x^m (i.e., the number of functions $\{1,2,...,m\} \to \{1,...,x\}$.

If every integer is covered by at least one marker, then one must count **<u>surjective</u>** functions $\{1,2,...,m\} \to \{1,...,x\}$. This is given by inclusion-exclusion as:

$$x^m - \binom{x}{1}(x-1)^m + \binom{x}{2}(x-2)^m - \bullet\bullet\bullet .$$

Now let: $\mathbf{P(x) = 1 - \frac{Number\ of\ surjective\ functions}{Number\ of\ all\ functions}}$

$$= \frac{\mathbf{x^m - \binom{x}{1}(x-1)^m + \binom{x}{2}(x-2)^m - \bullet\bullet\bullet .}}{\mathbf{x^m}} .$$

Then $\mathbf{P(x) \to 0}$ as $\mathbf{x \to \infty}$ (remembering $\mathbf{m \approx \frac{x}{\ln x}}$.)

When we count surjective functions, we leave "gaps" in the set of integers. Thus, there might be integers less than x not occurring as the average of two primes.

If $\mathbf{P^{\sim}(x) = \frac{No. of\ integers\ (< x\ not\ occurring\ as\ the\ average\ of\ 2\ primes)}{x^m}}$ then the problem is to prove $P^{\sim}(x) = 0$ for all x, rather than $P^{\sim}(x) \to 0$. That is, estimating $P^{\sim}(x)$ by a function *approaching* zero is not good enough.

The difficulty in proving the conjecture is that it is concerned with addition. Prime numbers are the multiplicative building blocks of the integers. The above effort is an attempt to show the conjecture true for a sequence of integers that are fairly dense in the sequence of integers with no very large gaps between its members. The advantage of focusing on the midpoints of primes is that the primes are simply not dense enough. We know we can find a sequence of n consecutive composites as large as we wish:

(n+1)! + 2, (n+1)! + 3, (n+1)! + 4, ..., (n+1)! + (n+1)

For example if n = 5, then five consecutive composites are: 722,723,724,725 and 726.

Thus, as we approach infinity, the gaps between the primes approach infinity very slowly. However, as is shown in the above summary table, the number of ways of writing an even integer as the sum of two primes (the number of midpoints) approaches infinity rapidly.

The probability that Goldbach's conjecture that every number greater than two is a sum of two primes is, **at least, almost certainly true.**